Universality of *A*-Mote Graphs

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UNIVERSALITY OF *A*-MOTE GRAPHS.

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In this note we consider only **finite loopless** graphs and digraphs. For graphs G and H, a **homomorphism** $G \to H$ is a mapping f of the vertex set of G to the vertex set of H such that if gg' is an edge of G, then $f(g)f(g')$ is an edge of H. The graph formed by the vertices $f(g)$ and the edges $f(g)f(g')$, over all vertices g and edges gg' of G, is called the **homomorphic image** of G. If A is a fixed graph, we say that a graph G is ***A*-mote** if there is no homomorphism $A \to G$. A graph is ***b*-bounded** if the degrees of all vertices are bounded by the integer b. If $\mathcal{C}$ is a class of graphs, we say that U is **universal for $\mathcal{C}$** (or $\mathcal{C}$ admits a universal graph U) if each graph G in $\mathcal{C}$ admits a homomorphism $G \to U$. (Note that U need not be a member of $\mathcal{C}$.)

We begin by remarking that the class of all b-bounded graphs does admit a universal graph, namely K_{b+1}. (This is another way of saying that any b-bounded graph admits a $b+1$ - colouring.) In this case the universal graph is itself b-bounded. We shall show that if we restrict the class of b-bounded graphs by requiring that they be A-mote (for a fixed connected graph A), we can find a universal graph which is also A-mote (although not necessarily b-bounded):

THEOREM. Let A be any fixed connected graph, and b a positive integer. The class of all A-mote b-bounded graphs admits a universal A-mote graph U.

Another way to view our result is the following: If a class of b-bounded graphs G can be characterized by the absence of homomorphisms **from** an arbitrary fixed connected graph A, then it can also be characterized by admitting homomorphisms **into** a suitable fixed graph U. Specifically, for any connected graph A there exists a graph U such that the following equality of sets holds (denoting by $\mathcal{C}_b$ the class of all b-bounded graphs) :

$$\{G \in \mathcal{C}_b : \text{there is no homomorphism } A \to G\} = \{G \in \mathcal{C}_b : \text{there is a homomorphism } G \to U\}.$$

This point of view was pursued in [NP 1978] and [Komárek 1983], where results of this type were seen as prototypes of "good characterization" (in the sense of Edmonds). It is shown in [NP 1978] that (with some simple exceptions A) the class of all A-mote graphs does not admit a universal A-mote graph. Therefore our assumption of being b-bounded is necessary. (For directed graphs, [Komárek 1983] gives interesting examples A for which universal digraphs do exist.)

Before presenting the general construction proving our Theorem, we give a simpler construction of a universal K_3-mote graph U for the class of **cubic** K_3-mote graphs: The vertices of U are the $4\binom{22}{4}$ ordered pairs $(\{a,b,c\},d)$, where a, b, c, d are different elements of $\{1, 2, ..., 22\}$. A vertex $(\{a,b,c\},d)$ is adjacent in U to all vertices $(\{a',b',c'\},d')$ such that the sets $\{a,b,c\}$ and $\{a',b',c'\}$ are disjoint and $d \in \{a',b',c'\}$, $d' \in \{a,b,c\}$. Note that if $(\{a,b,c\},d)$ is adjacent to both $(\{a',b',c'\},d')$ and $(\{a'',b'',c''\},d'')$ then $(\{a',b',c'\},d')$ and $(\{a'',b'',c''\},d'')$ are not adjacent to each other because the sets $\{a',b',c'\}$ and $\{a'',b'',c''\}$ both contain d. Therefore U contains no K_3 and hence is also K_3-mote. We now exhibit a homomorphism $f : G \to U$ for an arbitrary K_3-mote

graph G. First of all, the graph G^3 (obtained from G by joining all vertices of distance at most three in G) has maximum degree at most 21, since G is cubic. Hence G^3 admits a vertex colouring with the colours $\{1, 2, ..., 22\}$. Now each vertex of G has received a colour so that vertices within distance three have different colours. The homomorphism f assigns to each vertex g of G the ordered pair $f(g) = (\{a,b,c\},d)$ where d is the colour of g and $\{a,b,c\}$ are the colours of its three neighbours in G. If vertices g, g' are adjacent in G then they and all their neighbours have distinct colours, and the ordered pairs $f(g)$ and $f(g')$ are adjacent in U.

Proof of the Theorem. Assume that the radius of A is a, i.e., that there is a vertex v of A such that the maximum distance $d_A(x,v)$ over all vertices x of A is a. Let V be a fixed set of cardinality $1+b+b^2+ ... +b^{2a+2}$. The universal graph U will be defined as follows: The vertices of U are all rooted graphs (R,r) such that

(i) $V(R)$ is a subset of V

(ii) each vertex v of R has $d_R(v,r) \le a+1$

(iii) R is b-bounded and A-mote.

Adjacency in U is defined by the following rule: the vertices (X,x) and (Y,y) of U are adjacent provided

(iv) the set $Q = V(X) \cap V(Y)$ induces the same subgraph in X and in Y

(v) x and y are adjacent in X and in Y

(vi) any vertex v of X with $d_X(v,x) \le a$ belongs to Q

(vii) any vertex v of Y with $d_Y(v,y) \le a$ belongs to Q.

We first prove that U is A-mote.

Lemma 1. If (X_0,x_0), (X_1,x_1), ..., (X_t,x_t) is a walk in U, and if $t \le a+1$, then $x_0, x_1, ..., x_t$ is a walk in X_t.

Proof. We observe that when $t=1$ the adjacency of (X_0,x_0) and (X_1,x_1) in U implies by the definitions that x_0 and x_1 are adjacent in X_1. We complete the proof by induction on t : If (X_0,x_0), (X_1,x_1), ..., (X_t,x_t) is a walk in U, and if $t \le a+1$, then by the induction hypothesis $x_0, x_1, ..., x_{t-1}$ is a walk in X_{t-1}. It follows that all the vertices $x_0, x_1, ..., x_{t-1}$ have distance at most $t-1 \le a$ from x_{t-1} in X_{t-1}, and the same is also true of x_t, since (X_{t-1},x_{t-1}) and (X_t,x_t) are adjacent in U and so x_{t-1}, x_t are adjacent in X_{t-1}. It follows from (vi) that all the vertices $x_0, x_1, ..., x_t$ also belong to X_t and because of (iv) they form a walk in X_t.

In a graph H we let $D_H(h, j)$ denote the subgraph of H induced by the disk of radius j centered at h, i.e., by the set $\{v : d_H(v,h) \le j\}$.

Lemma 2. For every vertex (R,r) of U, there is a homomorphism

$$D_U((R,r), a) \to D_R(r, a) .$$

Proof. Consider the mapping that takes any (X,x) to x. If (X,x), (Y,y) are adjacent vertices of U, of distance at most a from (R,r), then there is a walk (X_0,x_0), (X_1,x_1), ..., (X_t,x_t) in U, with $X_0 = X$, $x_0=x$, $X_1 =Y$, $x_1=y$, $X_t =R$, $x_t=r$, and

$t \leq a+1$. By Lemma 1, $x_0, x_1, ..., x_t$ is a walk in X_t , and in particular x , y are adjacent in $X_t = R$. This argument also shows that $d_R(y,r) \leq a$; reversing the roles of x and y shows that also $d_R(x,r) \leq a$.

To prove that U is A-mote, we now assume that B is a homomorphic image of A and a subgraph of U . Clearly, B has radius at most a , and hence is a subgraph of some disk $D_U((R,r), a)$. The homomorphism of Lemma 2 implies that B , and hence also A , has a homomorphism into R , contrary to (iii).

Next we verify that U is universal for A-mote b-bounded graphs. Let G be any A-mote b-bounded graph, and let $p=2a+2$. The graph G^p has degree at most

$$b+b^2+ ... +b^p$$

and hence can be coloured with the set V . We obtain a mapping $f : V(G) \rightarrow V$ such that $d_G(x,y) \leq a+1$ implies $f(x) \neq f(y)$. Now we define a homomorphism $h : G \rightarrow U$ as follows: if g is a vertex of G then $h(g) = (R,r)$ where $r = f(g)$ and R is isomorphic to $D_G(g, a+1)$ via the mapping f . It is easy to see that R satisfies (i-iii) and that h is a homomorphism.

In the remainder of this note we derive some consequences of the above construction, make related observations, and pose some unsolved problems.

(A) First of all, we note that we may assume that the graph A does not admit a homomorphism into a proper subgraph A' -- otherwise we may deal with A'-mote graphs instead, as G is A-mote if and only if it is A'-mote. A graph A with the above property is sometimes called a **core** [HN 1989].

(B) Next we remark that our Theorem is in a sense best possible, as **disconnected** cores A do **not** have the property that the class of A-mote b-bounded graphs always admits a universal A-mote graph. Indeed, let A be a disconnected core with components $A_1, ..., A_n$ ($n>1$), and maximum degree at most b. Suppose U is an A-mote universal graph for the class of A-mote b-bounded graphs. Since A is a core, there is no homomorphism $A \to A_i$ for any i, i.e., each component A_i is an A-mote b-bounded graph and hence admits a homomorphism into U. But this means that U is not A-mote - a contradiction.

(C) By letting A be an odd cycle, we conclude from the above Theorem that the class of b-bounded graphs with odd girth at least g (i.e., without odd cycles of length less than g) admits a universal graph with odd girth at least g. Note that the class of **all** graphs with infinite odd girth (i.e., without any odd cycles) has a universal graph with infinite odd girth, namely K_2; we don't need here the restriction to b-bounded graphs. (This last observation amounts to saying that any bipartite graph admits a homomorphism to K_2.)

(D) For **digraphs**, we have a similar result: Let A be any fixed connected digraph. The class of all A-mote b-bounded digraphs admits a universal A-mote graph U.

Here a homomorphism f of **digraphs** G to H is defined to map an **arc** gg' of G to an **arc** $f(g)f(g')$ of H. A digraph is **connected**, respectively ***b*-bounded**, if its underlying graph is connected, respectively b-bounded. The proof proceeds exactly as in the previous case, with the distances in a digraph being taken in the corresponding underlying graph.

(E) In the special case when A is a directed two-cycle, A-mote digraphs are just **oriented** graphs. **(D)** asserts that the class of b-bounded oriented graphs admits a universal oriented graph U_b. Below we give a somewhat simpler construction for this special case (similar to the first example given for the cubic K_3-mote graphs):

The vertices of U_b are triples (j,X,Y) such that j is an integer from the set $S = \{0, 1, ..., b^2\}$, X is a subset of $S-\{j\}$ with at most b elements, and Y is a subset of X. The pair (j,X,Y) (k,Z,W) is an arc of U_b just if $k \in Y$ and $j \in Z-W$. It is immediate from the definition that U_b is an oriented graph. Given any b-bounded oriented graph G, we can again colour the vertices of G with colours from S so that vertices of distance (in the graph underlying G) at most two obtain different colours. This allows us to define a homomorphism of G to U_b by mapping the vertex g of G to the triple (j,X,Y) where j is the colour of g, X the set of colours used on vertices adjacent to g in the underlying graph of G, and Y the set of colours used on the vertices dominated by g in G. It is easy to verify that this is indeed a homomorphism.

(F) Given a family $\mathcal{A}$ of connected graphs (or digraphs), we say that G is $\mathcal{A}$-mote if it is A-mote for all A in $\mathcal{A}$. It is not difficult to see how our results would extend to $\mathcal{A}$-mote graphs (digraphs). We only need to assume that the radius in $\mathcal{A}$ is bounded; the proofs apply almost verbatim.

We conclude with a few natural open problems:

(1) What is the minimum $f_A(b)$ such that the class of all A-mote b-bounded graphs

admits an A-mote $f_A(b)$-bounded universal graph ? If A is the directed two-cycle then clearly $f_A(2) > 2$ as can be seen by considering directed cycles of prime order, cf. [HHMN 1988]. Could $f_A(b)$ equal $b+1$?

(2) Is every large regular tournament universal for the class of oriented b-bounded digraphs ?

(3) What is the minimum order of an A-mote graph (digraph) universal for the class of A-mote b-bounded graphs (digraphs) ?

(4) What is the minimum chromatic number of an A-mote graph (digraph) universal for the class of A-mote b-bounded graphs (digraphs) ?

(5) Does the class of ***A*-free** b-bounded graphs admit a universal A-free graph ? (A graph is A-free if it has no subgraph isomorphic to A.) Is this so when A is the undirected four-cycle ? In a similar vein, does the class of b-bounded graphs with **girth** greater than g admit a universal graph ? What if $g = 4$?

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